
Lesson 09

# Rigid body Collisions and Joints

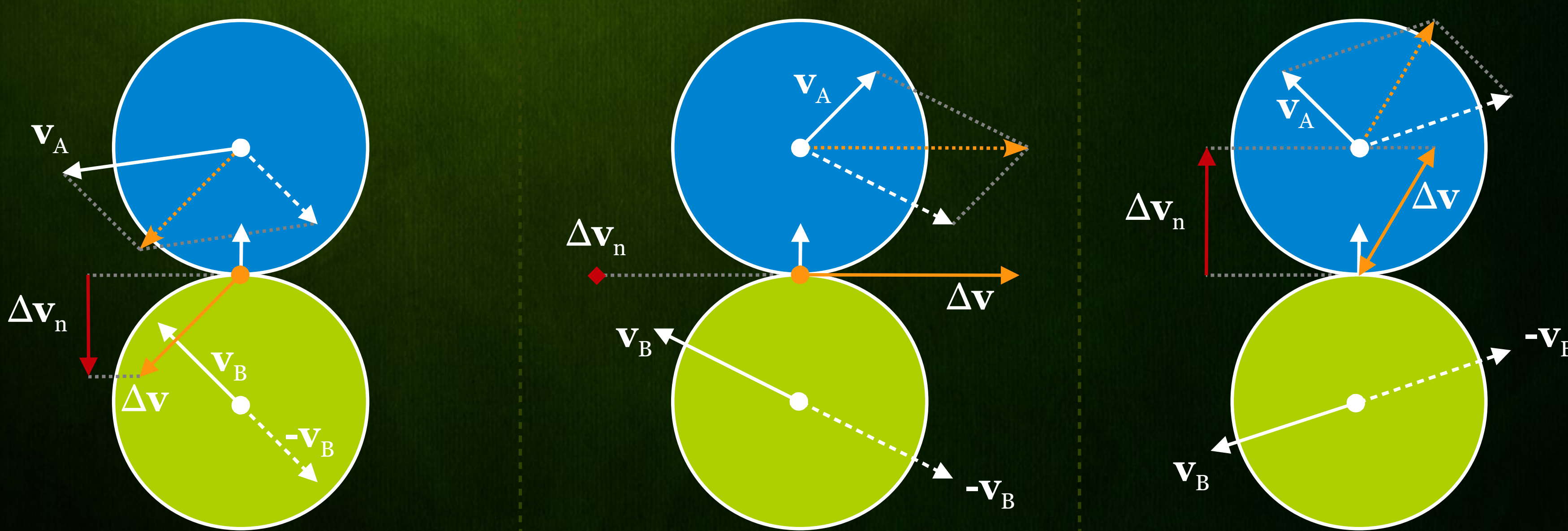
# Lesson 09 Outline

- Problem definition and motivations
- Simplified collision model
- Impulse based collision resolution
  - Friction-less collision resolution
  - Algebraic collision resolution for Coulomb friction
- Linear and angular joint formulations
- Demos / tools / libs

# Simplified collision model

# Contact Types

- Bodies either collide, rest or separate depending on their relative velocity of contact points
  - Assuming no rotational motion all 3 collision scenarios are:




Colliding Contact $\Delta v_n < 0$ | Resting Contact $\Delta v_n = 0$ | Separating contact $\Delta v_n > 0$

# Simplified collision model

- Perfect rigidity
  - Bodies are perfectly rigid. There are no plastic or elastic deformations, where kinetic energy is dissipated. Thus our impact models must artificially decrease the kinetic energy
- Very short collision interval
  - We model highly elastic behavior, making the collision interval $\Delta t$ very short requiring the repulsive forces to be very strong, to maintain the non-penetration constraint.
- Direct velocity change
  - We need to integrate response forces during the collision interval into impulses and change objects velocities directly, causing discontinuities of motion.

# Simplified collision model

- Non-impulsive forces are ignored
  - We can neglect all non-impulsive forces (e.g. gravity), because they are too small compared to the impulsive forces and have no time to accumulate during collision
- Point contact
  - We reduce the contact region to a set of point contacts treated either as a sequence of single collisions or as a simultaneous multiple impact similar to resting contact
- Constant state
  - We assume position, orientation, inertia tensor, contact point and contact normal constant, since their change during the collision is negligible. Velocities change strongly


Impulse based Collision Resolution

# Collision Resolution

- Rigid body collision resolution is described as Collision Laws composed of
- Impact Model
  - Describes rules which preserve the non-penetration constraints of colliding bodies
- Friction Model - is responsible for creating frictional effects as
  - Sticking – bodies rest on each other due to friction forces
  - Rolling – bodies start to roll due to friction forces
  - Sliding – bodies slow down sliding due to friction forces

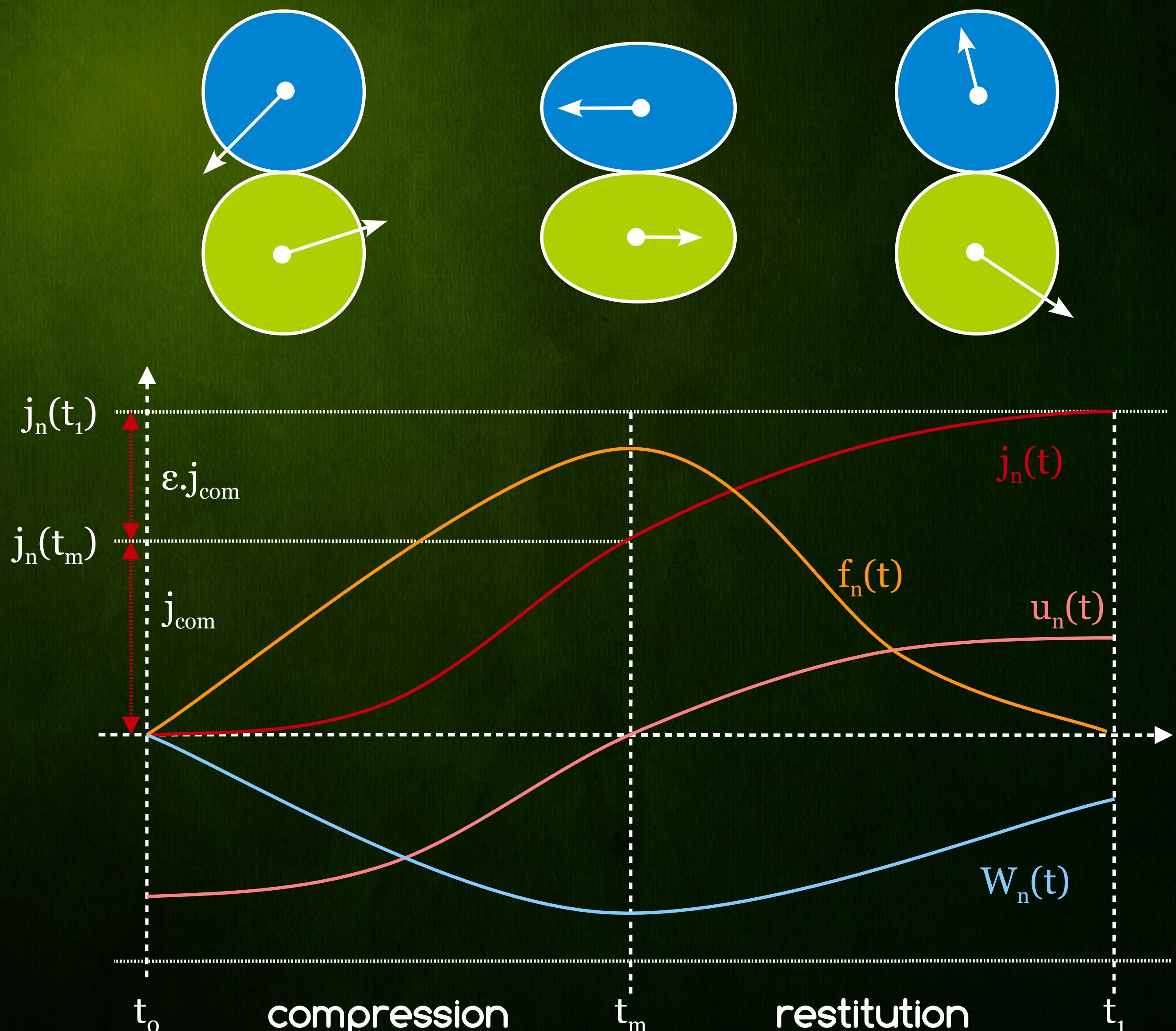
# Collision Resolution Strategies

- Algebraic Collision Resolution
  - Final velocities (impulse) are calculated using only algebraic relations between pre and post collision variables (velocities, energies… ). No numerical ODE solvers → fast
- Incremental Collision Resolution
  - Evolution of the impulsive forces are described with some (ordinary) differential equation with initial and final conditions formed for compression and restitution phases.
- Full Deformation Collision Resolution
  - Most accurate collision laws accounting with subtle stress and strain processes during the impact. Usually solved using finite element methods. Slow, not suitable for real-time apps.

# Impact Model

- In real world objects are never perfectly rigid.
  - First, their shape is **compressed.**
  - If they are elastic their shape is then **restituted.**
  - If they are plastic their shape is then plasticlly **deformed.**
- Impact model as a part of some collision law
  - Determines the post-collision velocities (positions, orientations… ) which prevent bodies to penetrate.
  - Models as realistic as possible the process during the compression and restitution.
- Time of maximum compression ($t_m$)
  - Time when compression ends and restitution starts.
  - Time when repulsive forces have maximal length

# Impact Model



# Newton's Impact Model

- Newton's Impact Model states simple algebraic linear relation between
  - Pre-collision relative normal velocity $u_n(t_0)$
  - Post-collision relative normal velocity $u_n(t)$
  - Based on coefficient of restitution $\varepsilon_n$
- Formally: $u_n(t) = -\varepsilon_n u_n(t_0) \quad \equiv \quad \mathbf{n}^T\mathbf{u}(t) = -\varepsilon_n \mathbf{n}^T\mathbf{u}(t_0)$
- Main drawbacks
  - it "blindly" finds some impulse, which cancels the relative velocity, but have no idea about restitution force accumulation during the compression and restitution phase
  - Can add kinetic energy during collision.

# Other Impact Models

- Poisson's Impact Model
  - Total impulse applied during compression $j_n(t_m)$ is proportional to the impulse applied during restitution $j_n(t1) - j_n(t_m)$
  - Formally: $j_n(t_1) - j_n(t_m) = \varepsilon_n j_n(t_m)$
  - In friction-less case it is equal to Newton's model
- Stronge's Impact Model
  - Directly relates the work of repulsive forces during compression $W_n(t_m)$ and restitution $W_n(t_1)$ - $W_n(t_m)$
  - Formally: $W_n(t_1) - W_n(t_m) = -\varepsilon^2{}_n W_n(t_m)$
  - Kinetic energy can not be increased
  - Coefficient of normal restitution $\varepsilon_n$ is a property of material.

# Coulomb Friction Model

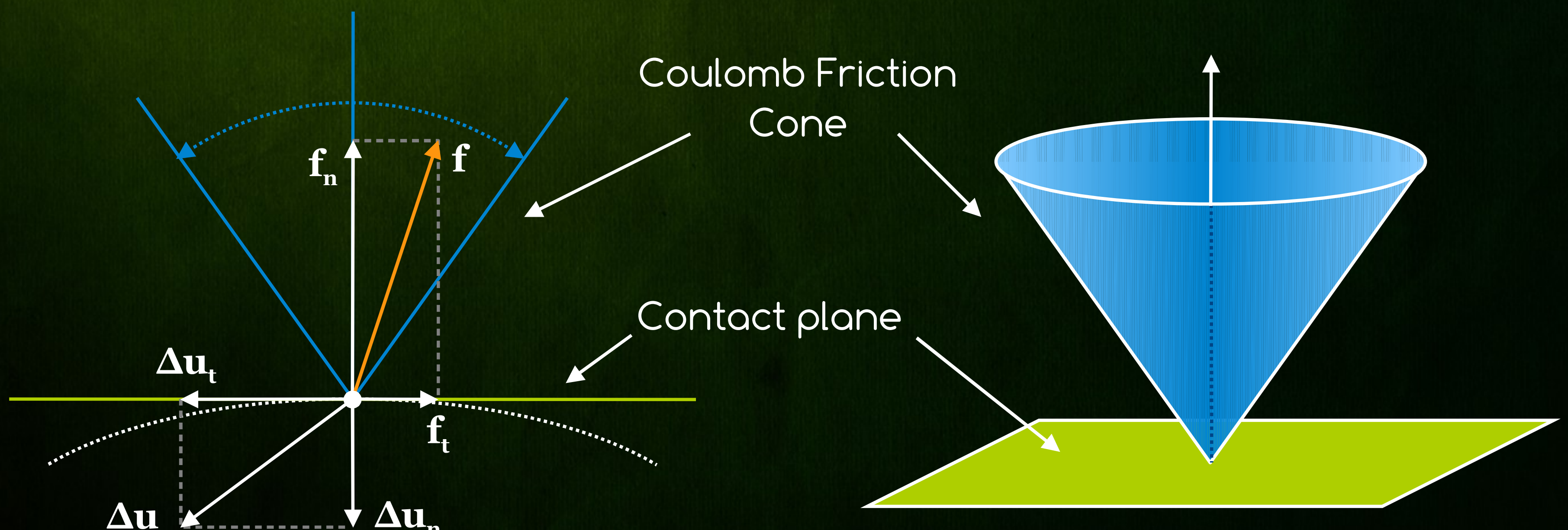
- In the real-world, microscopic interaction between colliding surfaces exerts frictional forces.
  - This process depends on many different factors, as microscopic structure of the surfaces, relative velocity, contact geometry, and other material properties.
- Assume **f** is the repulsive force between bodies acting on contact point **p** and **u** is relative velocity
- Both **f** and **u** can be split into
  - Normal components ($\mathbf{f}_n$, $\mathbf{u}_n$) parallel to contact normal
  - Tangential components ($\mathbf{f}_t$, $\mathbf{u}_t$) being inside contact plane
- $\mathbf{f} = \mathbf{f}_n + \mathbf{f}_t$ and $\mathbf{u} = \mathbf{u}_n + \mathbf{u}_t$

# Coulomb Friction Model

- Coulomb Friction Law
  - Friction force has opposite direction to relative tangential velocity and is proportional to normal repulsive force.
  - If the relative tangential velocity vanishes (is zero), we know only that the length of frictional component is less than μ times to the normal component.
  - μ is the **coefficient of friction** and depends only on material
- Sliding: $u_t \mathrel{!=} 0 \rightarrow f_t = -\mu|f_n|u_t / |u_t| \rightarrow |f_t| = \mu|f_n|$
- Sticking: $u_t == 0 \rightarrow |f_t| \leq \mu|f_n|$
- In both cases $|f_t(t)| \leq \mu|f_n(t)|$ thus for any direction friction force must lie in the friction cone

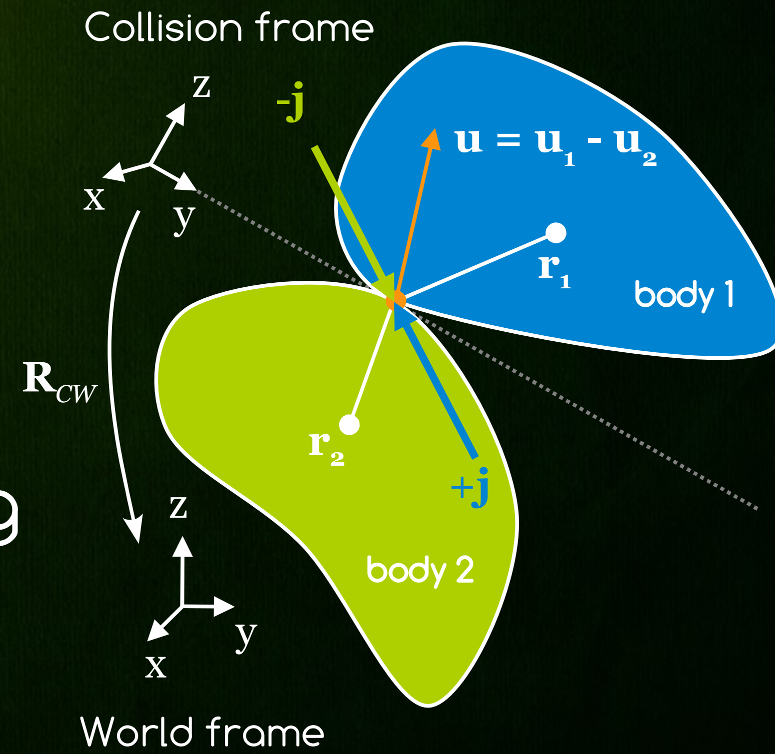
# Coulomb Friction Model

- Similar relation $|j_t| \le \mu|j_n|$ holds for impulses
  - $|j_t| = |\int^{t}_{t0} \mathbf{f}_t(\lambda)d\lambda| \le \int^{t}_{t0} |\mathbf{f}_t(\lambda)|d\lambda \le \mu\int^{t}_{t0} |\mathbf{f}_t(\lambda)|d\lambda = \mu|j_n|$



# Impulse base Collision Scenario

- Collision Frame
  - Origin is the contact point
  - Z axis is the contact normal
- Relative velocity u on contact point is: $u = u_1 - u_2$
- Local body positions of contact point are: $r_1$ and $r_2$
- Velocities are changed during collision due to applying collision impulses (+j) and (-j)



# Collision Impulse

- Collision Impulse j is the time integral of the repulsive force f over the collision interval $(t_0, t)$
  - $j = j(t) := \int^{t}_{to} f(\lambda) d\lambda$
- We define a delta operator "Δ" which for a given function "Ω" calculates the integral of its time derivative Ω' (= dΩ/dt) over collision interval $(t_0, t)$
  - $\Delta(\Omega) := \int^{t}_{to} \Omega'(\lambda) d\lambda = \Omega(t) - \Omega(t_0)$
- Due to Newton's Third (action-reaction) Law during the collision there are finite (but huge) repulsive forces which together with the opposite reactive forces are pushing bodies apart

# Collision Impulse

- Suppose some repulsive force $+f$ $(-f)$ pushes first (second) body at contact point $\rho$

- We can express f using Newton-Euler equation

$$(+f) = P'_1 = (M_1 v_1)' \qquad r_1 \times (+f) = L'_1 = (J_1 \omega_1)'$$

$$(-f) = P'_2 = (M_2 v_2)' \qquad r_2 \times (-f) = L'_2 = (J_2 \omega_2)'$$

- Using the "$\Delta$" operator we can express impulse $j$

$$(+j) = \Delta P_1 = M_1 \Delta v_1 \qquad r_1 \times (+j) = \Delta L_1 = J_1 \Delta \omega_1$$

$$(-j) = \Delta P_2 = M_2 \Delta v_2 \qquad r_2 \times (-j) = \Delta L_2 = J_2 \Delta \omega_2$$

# Collision Impulse

- The velocity change due to applying an impulse is

$\Delta v_1 = M_1^{-1}(+j)$ $\qquad$ $\Delta\omega_1 = J_1^{-1}(r_1 \times (+j))$

$\Delta v_2 = M_2^{-1}(-j)$ $\qquad$ $\Delta\omega_2 = J_2^{-1}(r_2 \times (-j))$

- If we express current velocities $u_1$, $u_2$ and their "change" $\Delta u_1$, $\Delta u_2$ at the contact point $\rho(t)$

$u_1 = v_1 + \omega_1 \times r_1$ $\qquad$ $\Delta u_1 = \Delta v_1 + \Delta\omega_1 \times r_1$

$u_2 = v_2 + \omega_2 \times r_2$ $\qquad$ $\Delta u_2 = \Delta v_2 + \Delta\omega_2 \times r_2$

# Collision Impulse

- The final "change" of velocities after the collision
  - $\Delta u_1 = M_1^{-1}(+j) + J_1^{-1}(r_1 \times (+j)) \times r_1 = \ldots = (M_1^{-1}\mathbf{1} + r_1^{\times} J_1^{-1} r_1^{\times})(+j) = K_1(+j)$
  - $\Delta u_2 = M_2^{-1}(-j) + J_2^{-1}(r_2 \times (-j)) \times r_2 = \ldots = (M_2^{-1}\mathbf{1} + r_2^{\times} J_2^{-1} r_2^{\times})(-j) = K_2(-j)$
- Final impulse-based collision equation is
- $\Delta u = \Delta u_1 - \Delta u_2 = K_1(+j) - K_2(-j) = (K_1 + K_2)\, j = K\, j(t)$
  - $K_1$ and $K_2$ are "Collision Matrices" of body 1 and 2
  - K is "Relative Collision Matrix" - symmetric positive definite
- Impulse-momentum equation is thus
- $j = K^{-1} \Delta u = K^{-1}(u(t) - u(t_0))$
- $u(t) = u(t0) + K\, j(t)$

# Friction-less Collision Resolution

- Using Newton's impact model collision impulse is
  - $\mathbf{K}\mathbf{j} = \Delta\mathbf{u} = \mathbf{u}(t) - \mathbf{u}(t_0)$ and $\mathbf{j} = |\mathbf{j}|\,\mathbf{j}_{\tilde{}}$
  - $\mathbf{n}^T\mathbf{K}|\mathbf{j}|\mathbf{j}_{\tilde{}} = \mathbf{n}^T\mathbf{u}(t) - \mathbf{n}^T\mathbf{u}(t_0) = -\varepsilon_n\mathbf{n}^T\mathbf{u}(t_0) - \mathbf{n}^T\mathbf{u}(t_0) = -(1 + \varepsilon_n)\mathbf{n}^T\mathbf{u}(t_0)$
  - $|\mathbf{j}| = -(1 + \varepsilon_n)\mathbf{n}^T\mathbf{u}(t_0) \,/\, \mathbf{n}^T\mathbf{K}\mathbf{j}_{\tilde{}}$
  - $\mathbf{j}_{\tilde{}}$ is unit direction vector of impulse (parallel with impulse)
- Collision impulse is related to pre-collision velocity
  - In friction-less case repulsive forces acts only in the normal direction (to stop penetration), thus impulse is parallel to contact normal: $\mathbf{j}_{\tilde{}}(t) = \mathbf{n}$
- $$\mathbf{j}(t) = |\mathbf{j}(t)|\mathbf{n} = \frac{-(1+\epsilon_n)\mathbf{n}^T\mathbf{u}(t_0)}{\mathbf{n}^T\mathbf{K}\mathbf{n}}\mathbf{n}$$

# Collision Resolution with Friction

- Considering friction we don't know the direction of the impulse.
- Any collision impulse must be admissible
  - It must preserve non-penetration, satisfy the friction cone condition and dissipate energy
- Friction cone Test
  - $j(t) = j_n(t) + j_t(t)$ and $j_n(t) = \mathbf{n}^T j(t)\mathbf{n}$
  - $|j(t) - \mathbf{n}^T j(t)\mathbf{n}| = |j_t(t)| \leq \mu |j_t(t)| = \mathbf{n}^T j(t)$
- $\text{test}(j) = |j - \mathbf{n}^T j\, \mathbf{n}| - \mathbf{n}^T j(t)$
  - If $\text{test}(j) \leq 0 \rightarrow$ impulse is in friction cone
  - If $\text{test}(j) > 0 \rightarrow$ impulse is not in friction cone

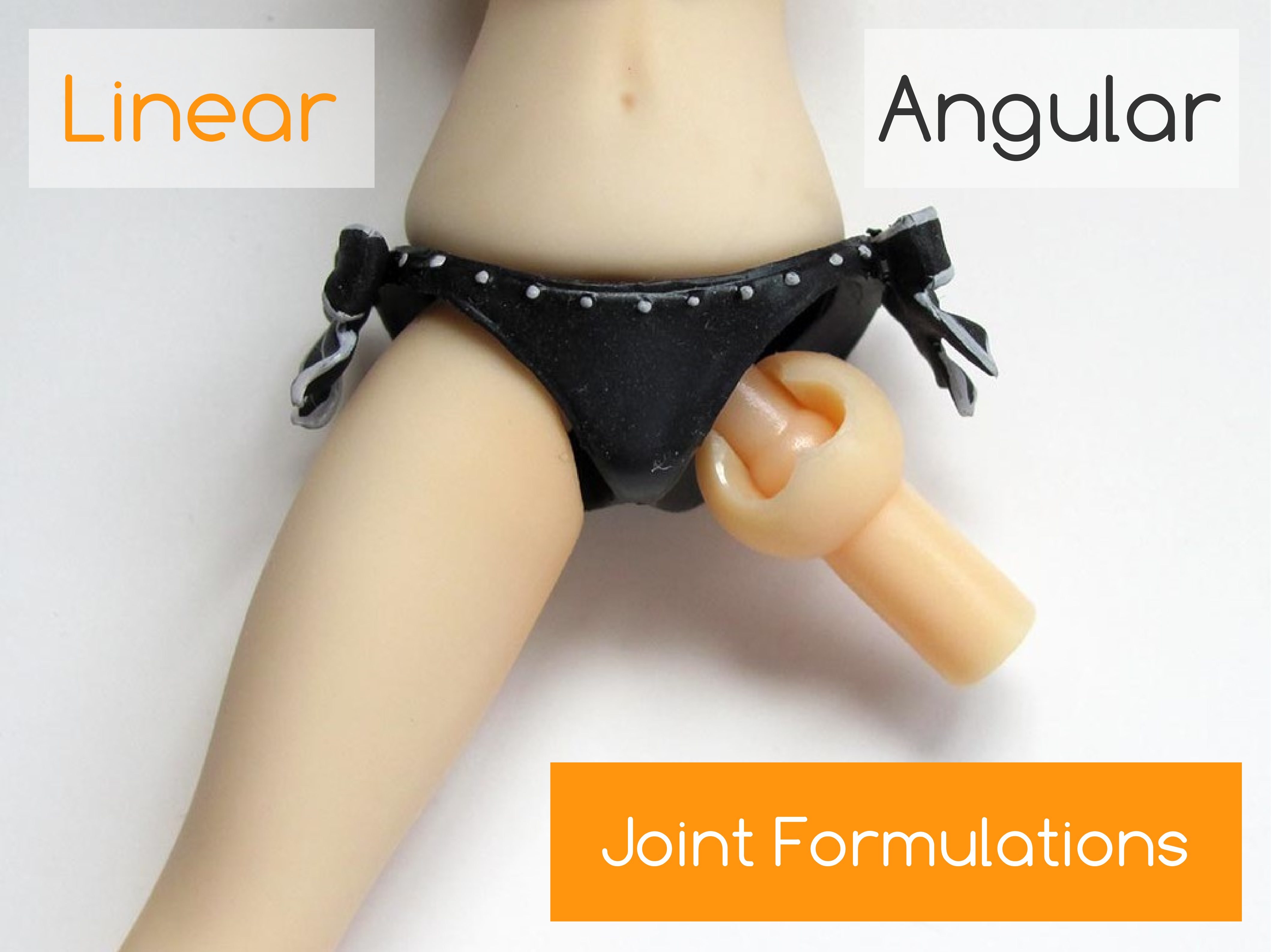
# Algebraic Resolution Law I

- Given some positive real c and any vectors A, B we define "projection" function "kappa" as

$$\mathrm{kappa}(c,\mathbf{A},\mathbf{B}) = \frac{c\mu\mathbf{n}^{\mathrm{T}}\mathbf{A}}{|\mathbf{B}-\mathbf{n}^{\mathrm{T}}\mathbf{B}\mathbf{n}|+\mu\mathbf{n}^{\mathrm{T}}(\mathbf{B}-\mathbf{A})}$$

- We define impulses $P_{\mathrm{I}}$ $P_{\mathrm{II}}$ and P
  - Plastic sliding $$\mathbf{P}_I = \frac{-(1+\epsilon_n)\mathbf{n}^{\mathrm{T}}\mathbf{u}(t_0)}{\mathbf{n}^{\mathrm{T}}\mathbf{K}\mathbf{n}}\mathbf{n} = \frac{-\mathbf{n}^{\mathrm{T}}\mathbf{u}(t_0)}{\mathbf{n}^{\mathrm{T}}\mathbf{K}\mathbf{n}}\mathbf{n}$$
  - Plastic sticking $$\mathbf{P}_{II} = \mathbf{K}^{-1}\left(\mathbf{u}(t)-\mathbf{u}(t_0)\right) = -\mathbf{K}^{-1}\mathbf{u}(t_0)$$
  - Predicted impulse $$\mathbf{P} = (1+\epsilon_n)\mathbf{P}_I+(1+\epsilon_t)\left(\mathbf{P}_{II}-\mathbf{P}_I\right)$$
- Final impulse is

$$\mathbf{j} = (1+\epsilon_n)\mathbf{P}_I+\kappa\left(\mathbf{P}_{II}-\mathbf{P}_I\right) \qquad \kappa = \begin{cases} (1+\epsilon_t) & \mathrm{test}(\mathbf{P})\leqslant 0 \\ \mathrm{kappa}(1+\epsilon_n,\mathbf{P}_I,\mathbf{P}_{II}) & \mathrm{test}(\mathbf{P})>0 \end{cases}$$

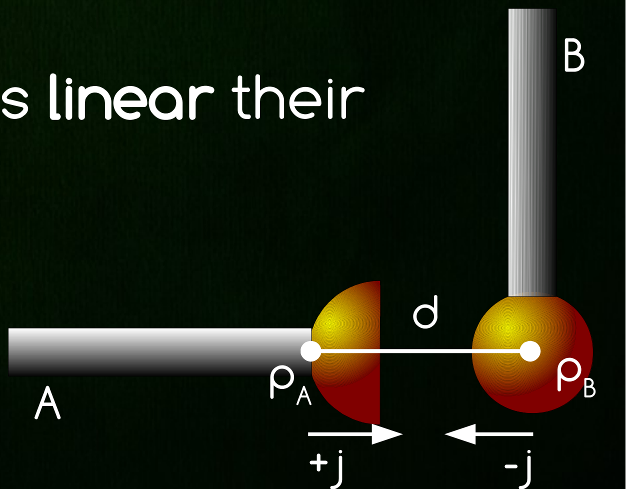

Linear
Angular
Joint Formulations

# Linear and Angular Joints

- 3 basic types of Linear joints
  - 0,1,2,3 DOF for relative linear motion
  - Angular motion is unconstrained (= 3 angular DOF)
- 3 basic types of Angular joints
  - 0,1,2,3 DOF for relative angular motion
  - Linear motion is unconstrained (= 3 linear DOF)
- Any 0-6 DOF joint constraint can be constructed as a combination of one linear and one angular joint
  - Ball Joint = 0 linear and 3 angular DOF (= 3 DOF)
  - Hinge Joint = 0 linear and 1 angular DOF (= 1 DOF)
  - Point on Plane Joint = 2 linear and 3 angular DOF (= 5 DOF)
  - Other joints ...

# 0-DOF Linear Joint

- 0 linear DOF = Relative linear motion of bodies is fully constrained at some joint point $\rho$
  - Let $\rho_A$ and $\rho_B$ be on bodies A and B where the joint is applied.
- To satisfy this joint, distance between $\rho_A$ and $\rho_B$ should be zero (within tolerance): $|\rho_A - \rho_B| \rightarrow 0$
  - Suppose at $t_0$ the joint is satisfied. After $\Delta t$ of free motion distance $d = \rho_A - \rho_B$ can become non-zero.
  - Simplifying the relative motion of $\rho_A$ and $\rho_B$ is **linear** their relative velocity is simply $\Delta u = d / \Delta t$
- From Impulse-momentum equation
- $j = K^{-1} \Delta u = K^{-1} (d / \Delta t)$



# 1-DOF Linear Joint

- 1 linear DOF = Relative linear motion of bodies is allowed along some line defined in one body
  - Let $\mathbf{l}_A = (c_A, a_A)$ be the allowed line on A and $p_B$ joint point on B
- To satisfy this joint distance between $l_A$ and $p_B$ should be zero: $d(l_A, p_B) \rightarrow 0$
- Similarly to previous joint we find the distance vector d between $l_A$ and $p_B$ and compute impulse
- $j = K^{-1}\Delta u = K^{-1}(d / \Delta t)$

# 2-DOF Linear Joint

- 2 linear DOF = Relative linear motion of bodies is allowed along some plane defined in one body
  - Let $\beta_A = (c_A, n_A)$ be the allowed plane on A; $\rho_B$ joint point on B
- To satisfy this joint distance between $\beta_A$ and $\rho_B$ should be zero: $d(\beta_A, \rho_B) \rightarrow 0$
- Similarly to previous joint we find the distance vector d between $\beta_A$ and $\rho_B$ and compute impulse
- $j = K^{-1} \Delta u = K^{-1} (d / \Delta t)$

# 3-DOF Linear Joint

- 3 linear DOF = Relative linear motion of bodies is unconstrained.
- We do not need to apply any impulse here
  - Assuming 3 angular DOF, the proposed joint has all DOF → Both relative linear and angular motion of bodies is unconstrained → there is no constraint at all. Bodies can freely move.

# 0-DOF Angular Joint

- 0 angular DOF = Relative angular motion of bodies is fully constrained
  - Let $q_{A0}$ and $q_{B0}$ be initial orientation of A and B
  - Relative orientation of A and B is $\Delta q = (q^{-1}_{B0}\, q_B)^{-1}(q^{-1}_{A0}\, q_A)$
  - $\Delta q$ is converted into axis-angle notation $(a, \alpha)$
- To satisfy this joint relative orientation $\Delta q$ should be zero: $\Delta q \rightarrow 0$
  - If relative angular motion is linearized relative angular velocity $\omega = (\omega_A - \omega_B)$ is proportional to the angle $\alpha$ along direction a during $\Delta t$: $\omega = \alpha . a / \Delta t$
- Angular momentum change is: $\Delta L = (J^{-1}_1 + J^{-1}_2)^{-1}\, \omega$
  - Change angular momentums: $L_A$ += $+\Delta L$ and $L_B$ += $-\Delta L$

# 1-DOF Angular Joint

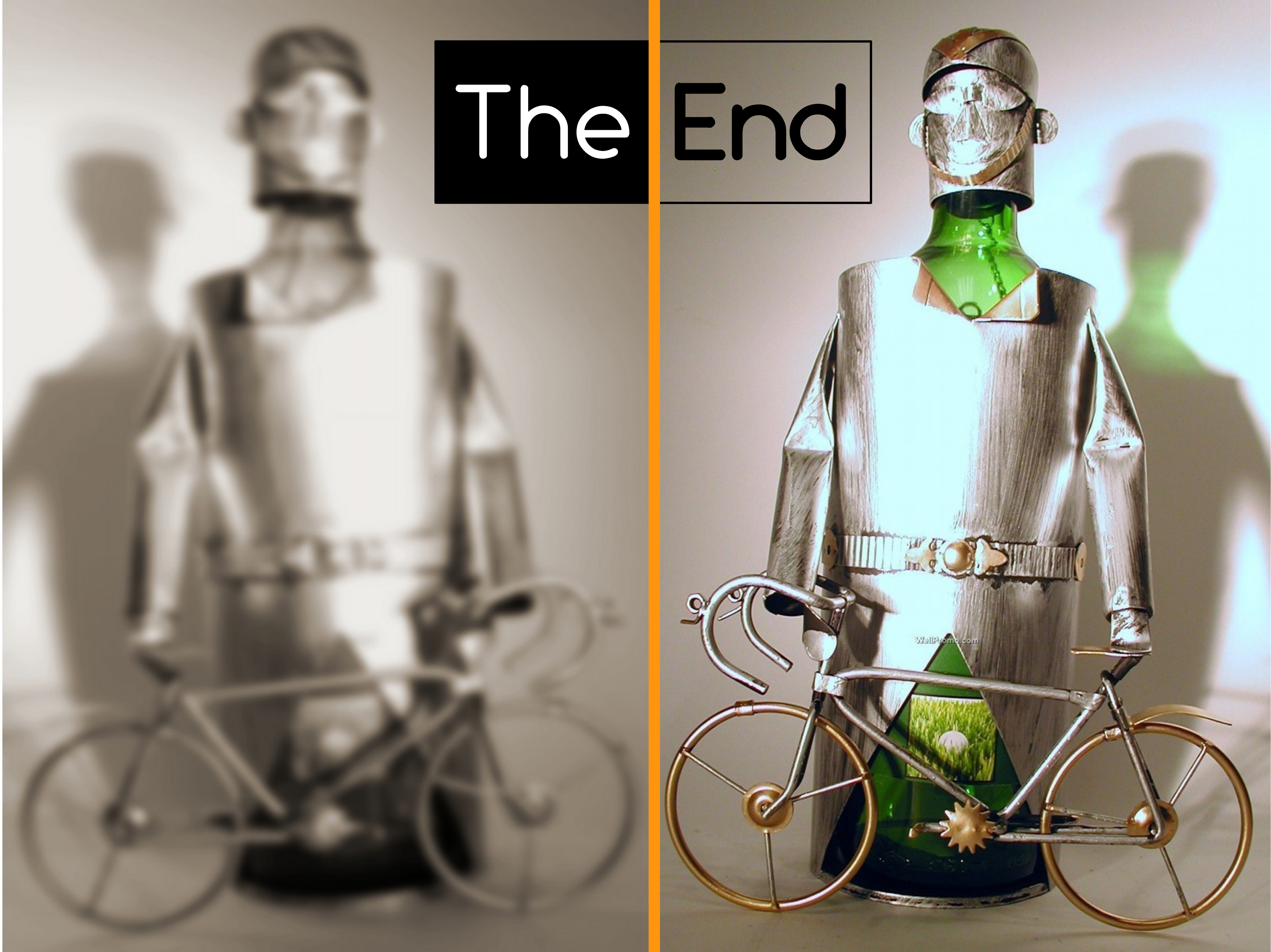
- 1 angular DOF = Bodies are allowed to rotate around one common axis (defined in both bodies)
  - Let $a_A$ and $a_B$ be the common unit axis in body A and B
  - Define the relative angular axis change as $d = a_A \times a_B$
  - Angular velocity change is proportional to d
- To satisfy this joint relative orientation change d should be zero: $d \rightarrow 0$
  - Similarly to previous joint relative angular velocity $\omega = d / \Delta t$
- Angular momentum change is: $\Delta L = (J^{-1}_1 + J^{-1}_2)^{-1} \omega$
  - Change angular momentums: $L_A$ += $+\Delta L$ and $L_B$ += $-\Delta L$

# 2-DOF Angular Joint

- 2 angular DOF = Bodies are allowed to rotate around two linearly independent axes.
  - Let $a_A$ and $b_B$ be unit rotation axes in body A and B
  - Define rotation change axis as $c = a_A \times b_B$
  - Angle $\varphi(t) = \arccos(a_A(t), b_B(t))$ between $a_A$ and $b_B$ must be constant during simulation
  - Relative orientation change is $d(t) = (\varphi(t) - \varphi(0))\, c$
- To satisfy this joint relative orientation change d should be zero: $d \rightarrow 0$
  - Similarly to previous joint, relative angular velocity $\omega = d / \Delta t$
- Angular momentum change is: $\Delta L = (J^{-1}_1 + J^{-1}_2)^{-1}\, \omega$
  - Change angular momentums: $L_A$ += $+\Delta L$ and $L_B$ += $-\Delta L$

# 3-DOF Angular Joint

- 3 angular DOF = Relative angular motion of bodies is unconstrained.
- We do not need to change angular momentum
  - Assuming 3 linear DOF, the proposed joint has all DOF → Both relative linear and angular motion of bodies is unconstrained → there is no constraint at all. Bodies can freely move.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