Lecture 3: Applications of FOL

2-AIN-108 Computational Logic

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Barber - Assignment

Proove that there is no male barber when

- every barber shaves beards of those men who do not shave their own beards
- no barber shaves beards of those men who shave their own beards

Barber - Alphabet

- predicate $p(x)$: x is a barber
- predicate $q(x, y)$: x shaves y's beard

Barber - Formalization

Theory T:

$$(\forall x)(p(x) \rightarrow (\forall y)(\neg q(y, y) \rightarrow q(x, y)))$$
$$(\forall x)(p(x) \rightarrow (\forall y)(q(y, y) \rightarrow \neg q(x, y)))$$

Formula ϕ:

$$\neg(\exists x)p(x)$$

Barber - Normal Form

Theory T:

$$\neg p(x) \lor q(y, y) \lor q(x, y)$$
$$\neg p(x) \lor \neg q(y, y) \lor \neg q(x, y)$$

Formula $\neg\phi$:

$$p(c)$$

1. $p(c)$ (assumption)

Barber - Reasoning

1. $p(c)$ (assumption)
2. $\neg p(x) \lor q(y, y) \lor q(x, y)$ (assumption)

Barber - Reasoning

1. $p(c)$ (assumption)
2. $\neg p(x) \vee q(y,y) \vee q(x,y)$ (assumption)
3. $\neg p(x) \vee \neg q(y,y) \vee \neg q(x,y)$ (assumption)

Barber - Reasoning

1. $p(c)$ (assumption)
2. $\neg p(x) \vee q(y, y) \vee q(x, y)$ (assumption)
3. $\neg p(x) \vee \neg q(y, y) \vee \neg q(x, y)$ (assumption)
4. $q(y, y) \vee q(c, y)$ (resolution of 1 and 2 with $\{x \mapsto c\}$)

Barber - Reasoning

1. $p(c)$ (assumption)
2. $\neg p(x) \lor q(y, y) \lor q(x, y)$ (assumption)
3. $\neg p(x) \lor \neg q(y, y) \lor \neg q(x, y)$ (assumption)
4. $q(y, y) \lor q(c, y)$ (resolution of 1 and 2 with $\{x \mapsto c\}$)
5. $\neg q(y, y) \lor \neg q(c, y)$ (resolution of 1 and 3 with $\{x \mapsto c\}$)

Barber - Reasoning

1. $p(c)$ (assumption)
2. $\neg p(x) \vee q(y,y) \vee q(x,y)$ (assumption)
3. $\neg p(x) \vee \neg q(y,y) \vee \neg q(x,y)$ (assumption)
4. $q(y,y) \vee q(c,y)$ (resolution of 1 and 2 with $\{x \mapsto c\}$)
5. $\neg q(y,y) \vee \neg q(c,y)$ (resolution of 1 and 3 with $\{x \mapsto c\}$)
6. $q(c,c)$ (factor of 4)

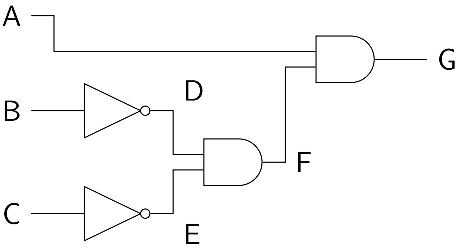
Barber - Reasoning

1. $p(c)$ (assumption)
2. $\neg p(x) \vee q(y, y) \vee q(x, y)$ (assumption)
3. $\neg p(x) \vee \neg q(y, y) \vee \neg q(x, y)$ (assumption)
4. $q(y, y) \vee q(c, y)$ (resolution of 1 and 2 with $\{x \mapsto c\}$)
5. $\neg q(y, y) \vee \neg q(c, y)$ (resolution of 1 and 3 with $\{x \mapsto c\}$)
6. $q(c, c)$ (factor of 4)
7. $\neg q(c, c)$ (factor of 5)

Barber - Reasoning

1. $p(c)$ (assumption)
2. $\neg p(x) \vee q(y, y) \vee q(x, y)$ (assumption)
3. $\neg p(x) \vee \neg q(y, y) \vee \neg q(x, y)$ (assumption)
4. $q(y, y) \vee q(c, y)$ (resolution of 1 and 2 with $\{x \mapsto c\}$)
5. $\neg q(y, y) \vee \neg q(c, y)$ (resolution of 1 and 3 with $\{x \mapsto c\}$)
6. $q(c, c)$ (factor of 4)
7. $\neg q(c, c)$ (factor of 5)
8. $\bot$ (resolution of 6 and 7 with $\{\}$)

Boolean Circuits - Assignment



If the value of A is 0, the value of B is 1, and the value of C is 1, what is the value of G?

Boolean Circuits - Alphabet

- constants a, b, c, d, e, f denote wires
- constants 0 and 1 refer to the boolean values that wire may possess
- predicate $val(w, v)$: the value of a wire w in the circuit is v
- predicate $and(x, y, z)$: there is an and-gate in the circuit that connects inputs x and y to an output z
- predicate $or(x, y, z)$: there is an or-gate in the circuit that connects inputs x and y to an output z
- predicate $inv(x, y)$: there is an inverter in the circuit that connects an input x to an output y

Boolean Circuits - Formalization

Theory T:

- Input: $inv(b, d) \quad inv(c, e) \quad and(d, e, f) \quad and(a, f, g)$
- AND gate

$$(\forall x)(\forall y)(\forall z)(and(x, y, z) \wedge (val(x, 1) \wedge val(y, 1)) \rightarrow val(z, 1))$$
$$(\forall x)(\forall y)(\forall z)(and(x, y, z) \wedge (val(x, 0) \vee val(y, 0)) \rightarrow val(z, 0))$$

- OR gate

$$(\forall x)(\forall y)(\forall z)(or(x, y, z) \wedge (val(x, 1) \vee val(y, 1)) \rightarrow val(z, 1))$$
$$(\forall x)(\forall y)(\forall z)(or(x, y, z) \wedge (val(x, 0) \wedge val(y, 0)) \rightarrow val(z, 0))$$

- NOT gate

$$(\forall x)(\forall y)(inv(x, y) \wedge val(x, 0) \rightarrow val(y, 1))$$
$$(\forall x)(\forall y)(inv(x, y) \wedge val(x, 1) \rightarrow val(y, 0))$$

Query ϕ: $val(a, 0) \wedge val(b, 1) \wedge val(c, 1) \rightarrow val(g, 0)$

Boolean Circuits - Normal Form

Theory T:

$$inv(b,d) \quad inv(c,e) \quad and(d,e,f) \quad and(a,f,g)$$

$$\neg and(x,y,z) \lor \neg val(x,1) \lor \neg val(y,1) \lor val(z,1)$$

$$\neg and(x,y,z) \lor \neg val(x,0) \lor val(z,0)$$

$$\neg and(x,y,z) \lor \neg val(y,0) \lor val(z,0)$$

$$\neg or(x,y,z) \lor \neg val(x,1) \lor val(z,1)$$

$$\neg or(x,y,z) \lor \neg val(y,1) \lor val(z,1)$$

$$\neg or(x,y,z) \lor \neg val(x,0) \lor \neg val(y,0) \lor val(z,0)$$

$$\neg inv(x,y) \lor \neg val(x,0) \lor val(y,1)$$

$$\neg inv(x,y) \lor \neg val(x,1) \lor val(y,0)$$

Formula $\neg\phi$:

$$val(a,0) \quad val(b,1) \quad val(c,1) \quad \neg val(g,0)$$

Boolean Circuits - Reasoning

1. $and(a, f, g)$ (assumption)

Boolean Circuits - Reasoning

1. $and(a, f, g)$ (assumption)
2. $\neg and(x, y, z) \vee \neg val(x, 0) \vee val(z, 0)$ (assumption)

Boolean Circuits - Reasoning

1. $and(a, f, g)$ (assumption)
2. $\neg and(x, y, z) \vee \neg val(x, 0) \vee val(z, 0)$ (assumption)
3. $val(a, 0)$ (assumption)

Boolean Circuits - Reasoning

1. $and(a, f, g)$ (assumption)
2. $\neg and(x, y, z) \vee \neg val(x, 0) \vee val(z, 0)$ (assumption)
3. $val(a, 0)$ (assumption)
4. $\neg val(g, 0)$ (assumption)

1. $and(a, f, g)$ (assumption)
2. $\neg and(x, y, z) \vee \neg val(x, 0) \vee val(z, 0)$ (assumption)
3. $val(a, 0)$ (assumption)
4. $\neg val(g, 0)$ (assumption)
5. $\neg val(a, 0) \vee val(g, 0)$ (resolution of 1 and 2 with $\{x \mapsto a, y \mapsto f, z \mapsto g\}$)

1. $and(a, f, g)$ (assumption)
2. $\neg and(x, y, z) \vee \neg val(x, 0) \vee val(z, 0)$ (assumption)
3. $val(a, 0)$ (assumption)
4. $\neg val(g, 0)$ (assumption)
5. $\neg val(a, 0) \vee val(g, 0)$ (resolution of 1 and 2 with $\{x \mapsto a, y \mapsto f, z \mapsto g\}$)
6. $\neg val(a, 0)$ (resolution of 4 and 5 with $\{\}$)

Boolean Circuits - Reasoning

1. $and(a, f, g)$ (assumption)
2. $\neg and(x, y, z) \vee \neg val(x, 0) \vee val(z, 0)$ (assumption)
3. $val(a, 0)$ (assumption)
4. $\neg val(g, 0)$ (assumption)
5. $\neg val(a, 0) \vee val(g, 0)$ (resolution of 1 and 2 with $\{x \mapsto a, y \mapsto f, z \mapsto g\}$)
6. $\neg val(a, 0)$ (resolution of 4 and 5 with $\{\}$)
7. $\bot$ (resolution of 3 and 6 with $\{\}$)