Computational Logic

Propositional Logic

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Alphabet

An *alphabet* contains

- Propositional variables
  $p, q, r, \ldots$
- Logical connectives
  $\neg, \vee, \wedge, \Rightarrow, \Leftrightarrow, \ldots$
- Punctuation symbols
  $(,)$

Formula

A *formula* is

- an atom (propositional variable)
- $\neg\Phi$ if Φ is a formula
- $(\Phi \wedge \Psi)$ if Φ and Ψ are formulas
- $(\Phi \vee \Psi)$ if Φ and Ψ are formulas
- $(\Pi \rightarrow \Psi)$ if Φ and Ψ are formulas
- $(\Pi \leftrightarrow \Psi)$ if Φ and Ψ are formulas
- ...

A *language* is a set $\mathcal{L}$ of all formulas.

Valuation

A *valuation* is a mapping $v : \mathcal{L} \mapsto \{0, 1\}$ such that

- $v(\neg\Phi) = 1$ iff $v(\Phi) = 0$
- $v(\Phi \wedge \Psi) = 1$ iff $v(\Phi) = 1$ and $v(\Psi) = 1$
- $v(\Phi \vee \Psi) = 1$ iff $v(\Phi) = 1$ or $v(\Psi) = 1$
- $v(\Phi \rightarrow \Psi) = 1$ iff $v(\Phi) = 0$ or $v(\Psi) = 1$
- $v(\Phi \leftrightarrow \Psi) = 1$ iff $v(\Phi) = v(\Psi)$

A formula Φ is *satisfiable* iff there exists a valuation v such that $v(\Phi) = 1$.
A formula Φ is a *tautology* iff for all valuations v holds $v(\Phi) = 1$.

Proposition

A formula Φ is not satisfiable iff $\neg\Phi$ is a tautology.

Entailment

A set of formulas T *entails* a formula Φ (denoted $T \models \Phi$) iff for all valuations v holds $v(\Phi) = 1$ whenever $v(\Psi) = 1$ for all Ψ in T.

Proposition

A formula Φ is a tautology iff $\emptyset \models \Phi$.

Proposition

$T \cup \{\Phi\} \models \Psi$ *iff* $T \models \Phi \rightarrow \Psi$.

A *inference rule* is an expression of the form

$$\frac{\Phi_1, \ldots, \Phi_n}{\Phi}$$

where $n \geq 0$, $\Phi_1, \ldots, \Phi_n$ and Φ are formulas. $\Phi_1, \ldots, \Phi_n$ are called *premises* and Φ is called the *conclusion*.

An *axiom* is an inference rule with empty set of premises.
A *propositional calculus* is a language with a set of inference rules.

A formula Φ is *provable* from a set of formulas T (denoted by $T \vdash \Phi$) iff there exists a sequence of formulas $\Phi_1, \ldots, \Phi_n = \Phi$, $n \geq 1$, such that for all formulas Φ_i, $1 \leq i \leq n$, holds

- $\Phi_i \in T$
- Φ_i is the conclusion of an inference rule with premises in $\{\Phi_1, \ldots, \Phi_{i-1}\}$

A propositional calculus is *sound* iff $\vdash \Phi$ implies $\models \Phi$.
A propositional calculus is *complete* iff $\models \Phi$ implies $\vdash \Phi$.

Hilbert System

Axioms

- $(P \to (Q \to P))$
- $((P \to (Q \to R)) \to ((P \to Q) \to (P \to R)))$
- $((\neg P \to \neg Q) \to (Q \to P))$

Inference Rules

- Modus Ponens

$$\frac{P, (P \to Q)}{Q}$$

Example

Prove:

$$(p \to p)$$

Proof:

1. $((p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p)))$
2. $(p \to ((p \to p) \to p))$
3. $((p \to (p \to p)) \to (p \to p))$
4. $(p \to (p \to p))$
5. $(p \to p)$

Sequent Calculus

A *sequent* is an expression of the form

$$\langle \Phi_1, \ldots, \Phi_m \Rightarrow \Psi_1, \ldots, \Psi_n \rangle$$

where $m \geq 0, n \geq 0$. $\Phi_1, \ldots, \Phi_n$ are called *antecendents* and $\Psi_1, \ldots, \Psi_n$ are called *succendents*.

A *inference rule* is an expression of the form

$$\frac{S_1, \ldots, S_n}{S}$$

where $n \geq 0$, $S_1, \ldots, S_n$ and S are sequents. $S_1, \ldots, S_n$ are called *premises* and S is called the *conclusion*.

An *axiom* is an inference rule with empty set of premises.
A *sequent calculus* is a language with a set of inference rules.

A formula Φ is *provable* from a set of formulas T (denoted by $T \vdash \Phi$) iff there exists a sequence of sequents $S_1, \ldots, S_n = \langle \Rightarrow \Phi \rangle$, $n \geq 1$, such that for all sequents S_i, $1 \leq i \leq n$, holds

- $S_i = \langle \Gamma \Rightarrow \Delta, \Phi \rangle$ for some $\Phi \in T$
- S_i is the conclusion of an inference rule with premises in $\{S_1, \ldots, S_{i-1}\}$

A sequent calculus is *sound* iff $\vdash \Phi$ implies $\models \Phi$.
A sequent calculus is *complete* iff $\models \Phi$ implies $\vdash \Phi$.

Gentzen System

- Axiom of Identity $\langle \Gamma, \phi \Rightarrow \Delta, \phi \rangle$
- Negation
  $\langle \Gamma \Rightarrow \Delta, \phi \rangle / \langle \Gamma, \neg\phi \Rightarrow \Delta \rangle$
  $\langle \Gamma, \phi \Rightarrow \Delta \rangle / \langle \Gamma \Rightarrow \Delta, \neg\phi \rangle$
- Conjunction
  $\langle \Gamma, \phi \Rightarrow \Delta \rangle / \langle \Gamma, \phi \wedge \psi \Rightarrow \Delta \rangle \quad \langle \Gamma, \phi \Rightarrow \Delta \rangle / \langle \Gamma, \psi \wedge \phi \Rightarrow \Delta \rangle$
  $\langle \Gamma \Rightarrow \Delta, \phi \rangle, \langle \Gamma \Rightarrow \Delta, \psi \rangle / \langle \Gamma \Rightarrow \Delta, \phi \wedge \psi \rangle$
- Disjunction
  $\langle \Gamma \Rightarrow \Delta, \phi \rangle / \langle \Gamma \Rightarrow \Delta, \phi \vee \psi \rangle \quad \langle \Gamma \Rightarrow \Delta, \phi \rangle / \langle \Gamma \Rightarrow \Delta, \psi \vee \phi \rangle$
  $\langle \Gamma, \phi \Rightarrow \Delta \rangle, \langle \Gamma, \psi \Rightarrow \Delta \rangle / \langle \Gamma, \phi \vee \psi \Rightarrow \Delta \rangle$
- Implication
  $\langle \Gamma, \phi \Rightarrow \Delta, \psi \rangle / \langle \Gamma \Rightarrow \Delta, \phi \rightarrow \psi \rangle$
  $\langle \Gamma \Rightarrow \Delta, \phi \rangle, \langle \Pi, \psi \Rightarrow \Lambda \rangle / \langle \Gamma, \Pi, \phi \rightarrow \psi \Rightarrow \Delta, \Lambda \rangle$
- Weakening Rule $\langle \Gamma \Rightarrow \Delta \rangle / \langle \Gamma \Rightarrow \Delta, \phi \rangle \quad \langle \Gamma \Rightarrow \Delta \rangle / \langle \Gamma, \phi \Rightarrow \Delta \rangle$
- Cut $\langle \Gamma \Rightarrow \Delta, \phi \rangle, \langle \Pi, \phi \Rightarrow \Lambda \rangle / \langle \Gamma, \Pi \Rightarrow \Delta, \Lambda \rangle$

Example

Prove:

$$p \vee \neg p$$

Proof:

1. $\langle p \Rightarrow p \rangle$
2. $\langle \Rightarrow p, \neg p \rangle$
3. $\langle \Rightarrow p, p \vee \neg p \rangle$
4. $\langle \Rightarrow p \vee \neg p \rangle$

Exercises

1. Extend the language of propositional logic with boolean constants $\top$ and $\bot$.
2. Decide if $\{(q \rightarrow r)\} \models ((p \rightarrow q) \rightarrow r)$.
3. Prove in Gentzen calculus $((p \rightarrow (q \vee r)) \rightarrow (((q \rightarrow \neg p) \wedge \neg r) \rightarrow \neg r))$.
   1. $\langle \Rightarrow ((p \rightarrow (q \vee r)) \rightarrow (((q \rightarrow \neg p) \wedge \neg r) \rightarrow \neg r)) \rangle$
   2. $\langle (p \rightarrow (q \vee r)) \Rightarrow (((q \rightarrow \neg p) \wedge \neg r) \rightarrow \neg r) \rangle$
   3. $\langle \Rightarrow ((q \rightarrow \neg p) \wedge \neg r) \rightarrow \neg r \rangle$
   4. $\langle ((q \rightarrow \neg p) \wedge \neg r) \Rightarrow \neg r \rangle$
   5. $\langle \neg r \Rightarrow \neg r \rangle$
4. Prove corectness of Hilbert axioms in Gentzen calculus.
5. Find a model for a set of formulas $\{p(c, d, d), (\forall x)(\forall y)(\forall z)(p(x, g(y), z) \rightarrow p(f(x), y, z)\}$ such that
   1. $p(x, y, z) \Leftrightarrow x + y = z$
   2. $p(x, y, z) \Leftrightarrow 2^x * y = z$
   3. $p(x, y, z) \Leftrightarrow |x| + y = z$