Lecture 6: Hypothetical Reasoning

2-AIN-144/2-IKV-131 Knowledge Reperesentation & Reasoning

Martin Baláž, Martin Homola

Department of Applied Informatics
Faculty of Mathematics, Physics and Informatics
Comenius University in Bratislava

21 Mar 2013

Deductive System

Definition (Inference Rule)

An inference rule r is an expression

$$\frac{\phi_1, \dots, \phi_m}{\phi_0}$$

where $0 \leq m$ and $\phi_0, \dots, \phi_m$ are well-formed formulas.

The formulas $pre(r) = \{\phi_1, \dots, \phi_m\}$ are called the *prerequisites* and the formula $cons(r) = \phi_0$ is called the *consequent* of r.

Definition (Deductive System)

A *deductive system* is a set of inference rules.

Proof

Definition (Provability)

Let R be a deductive system and T be a theory. A formula ϕ is *provable* from T in R (denoted $T \vdash_R \phi$) if there exists a sequence of inference rules $r_1, \ldots, r_n$, $0 \leq n$, in R such that

- $pre(r_i) \subseteq T \cup \{cons(r_1), \ldots, cons(r_{i-1})\}$ for each $0 < i \leq n$
- $\phi \in T \cup \{cons(r_1), \ldots, cons(r_n)\}$

Definition (Deductive Closure)

Let R be a deductive system and T be a theory. By $Cn_R(T)$ we denote the smallest theory T' which

- contains T, i.e. $T \subseteq T'$
- is *closed* under R, i.e. for each inference rule $r \in R$, if $pre(r) \subseteq T'$ then $cons(r) \in T'$.

Theorem

Let R be a deductive system and T be a theory in a language $\mathcal{L}$. Then $Cn_R(T) = \{\phi \in \mathcal{L} \mid T \vdash_R \phi\}$.

Birds usually fly (except penguins, ostriches, birds with boken wings, ...). Tweety is a bird.

Exceptions in Classical Logic

Birds usually fly (except penguins, ostriches, birds with boken wings, ...). Tweety is a bird.

Classical Logic:

$$\forall X(\mathit{bird}(X) \land \neg \mathit{penguin}(X) \land \neg \mathit{ostrich}(X) \land \\ \land \neg \mathit{broken\_wing}(X) \land \dots \supset \mathit{fly}(X))$$

$$\mathit{bird}(\mathit{tweety}) \land \neg \mathit{penguin}(\mathit{tweety}) \land \neg \mathit{ostrich}(\mathit{tweety}) \land \\ \land \neg \mathit{broken\_wing}(\mathit{tweety}) \land \dots$$

Default Inference Rules

Definition (Default)

A *default* is an expression d of the form

$$\frac{\phi_1, \ldots, \phi_m : \phi_{m+1}, \ldots, \phi_n}{\phi_0}$$

where $0 \leq m < n$ and $\phi_0, \ldots, \phi_n$ are well-formed formulas.

The formulas $pre(d) = \{\phi_1, \ldots, \phi_m\}$ are called the *prerequisites*, the formulas $just(d) = \{\phi_{m+1}, \ldots, \phi_n\}$ are called the *justifications*, and the formula $cons(d) = \phi_0$ is called the *consequence* of d.

The intuitive meaning of a default is the following:

> *If* $\phi_1, \ldots, \phi_m$ *are true, and we can assume* $\phi_{m+1}, \ldots, \phi_n$, *then* ϕ_0 *is true.*

Birds usually fly (except penguins, ostriches, birds with boken wings, ...). Tweety is a bird.

Default Logic:

$$\frac{bird(X) : fly(X)}{fly(X)}$$

$$\forall X(penguin(X) \supset \neg fly(X))$$
$$\forall X(ostrich(X) \supset \neg fly(X))$$
$$\forall X(bird(X) \wedge broken\_wing(X) \supset \neg fly(X))$$
$$\ldots$$
$$bird(tweety)$$

Closed World Assumption

If $R(\overline{x})$ is not derived ($\neg R(\overline{x})$ can be assumed), we conclude $\neg R(\overline{x})$.

$$\frac{: \neg employee(X)}{\neg employee(X)}$$

$$employee(peter)$$
$$employee(bob)$$

$$\neg employee(alice)?$$

Frame Problem (Yale Shooting Problem)

In dynamic worlds, what is not affected by actions, is unchanged.

$$\frac{holds(F,S), action(A) : holds(F, do(A,S))}{holds(F, do(A,S))}$$

$$\frac{\neg holds(F,S), action(A) : \neg holds(F, do(A,S))}{\neg holds(F, do(A,S))}$$

$$\forall S\ holds(loaded, do(load, S))$$

$$\forall S\ \neg holds(loaded, do(shoot, S))$$

$$\forall S\ \neg holds(alive, do(shoot, S))$$

$$action(load), action(wait), action(shoot)$$

$$holds(alive, s_0)$$

Richard Nixon is a Republican and at the same time a Quaker. Republicans are usually not Pacifists, but Quakers are.

$$\frac{republican(X) : \neg pacifist(X)}{\neg pacifist(X)}$$

$$\frac{quaker(X) : pacifist(X)}{pacifist(X)}$$

$$republican(nixon)$$

$$quaker(nixon)$$

Classical Logic is monotonic:

If $T \vdash \phi$ then $T' \vdash \phi$ for all $T \subseteq T'$.

Default Logic is nonmonotonic:

Consider a simple theory consisting of the single default $\frac{:a}{b}$. Thus b may be believed. If subsequently $\neg a$ is observed we then have the new theory in which b cannot be believed.

In nonmonotonic logics, beliefs can be revised in the presence of new information.

Default Theory

Definition (Default Theory)

A *default theory* is a pair $\Delta = (D, W)$ where

- D is a set of defaults
- W is a classical theory

We assume a sound and complete deductive system R.

We assume that all formulas in D and W are closed, i.e. they do not contain free variables. If they do, we substitute all ground terms for each free variable.

Contextual Proof

Definition (Contextual Provability)

Let $\Delta = (D, W)$ be a default theory and E be a theory. A formula ϕ is *provable* from Δ in the *context* E (denoted $W \vdash_{D(E)} \phi$) iff there exists a sequence of defaults $d_1, \ldots, d_n$, $0 \leq n$, such that

- $W \cup \{cons(d_1), \ldots, cons(d_{i-1})\} \vdash cons(d_i)$ for each $0 < i \leq n$
- $\overline{just(d_i)} \cap E = \emptyset$ for each $0 < i \leq n$
- $W \cup \{cons(d_1), \ldots, cons(d_n)\} \vdash \phi$

By $D(E)$ we denote the set of inference rules $\{pre(d)/cons(d) \mid d \in D, \overline{just(d)} \cap E = \emptyset\}$.

Reasoning in Default Theory

Definition (Operator Γ_Δ)

Let $\Delta = (D, W)$ be a default theory and E be a theory. By $\Gamma_\Delta(E)$ we denote the smallest theory E' which

- contains W, i.e. $W \subseteq E'$
- is closed under R, i.e. $Cn_R(E') = E'$
- is closed under $D(E)$, i.e. $Cn_{D(E)}(E') = E'$

Theorem

Let $\Delta = (D, W)$ be a default theory and E be a theory in a language $\mathcal{L}$. Then $\Gamma_\Delta(E) = \{\phi \in \mathcal{L} \mid W \vdash_{D(E)} \phi\}$.

Extensions of Default Theories

Definition (Default Extension)

Let Δ be a default theory. A theory E is a *default extension* of Δ iff $\Gamma_\Delta(E) = E$.

Example (No Extension)

Neither $E_1 = Cn_R(\emptyset)$ nor $E_2 = Cn_R(\{p\})$ is a default extension of $\Delta = (\{\frac{:\neg p}{p}\}, \emptyset)$.

Example (Many Extensions)

Both $E_1 = Cn_R(\{p\})$ and $E_2 = Cn_R(\{q\})$ are default extensions of $\Delta = (\{\frac{:\neg p}{q}, \frac{:\neg q}{p}\}, \emptyset)$.

Iterative Characterization of Extensions

Theorem

Let $\Delta = (D, W)$ *be a default theory and* E *be a theory. Define*

$$E_0 = W$$
$$E_{i+1} = Cn_R(E_i) \cup \{cons(d) \mid d \in D(E), pre(d) \subseteq E_i\}$$

Then E *is an extension of* Δ *iff* $E = \bigcup_{i=0}^{\infty} E_i$.

Proof.

We can observe that

(D1) $W \subseteq \bigcup_{i=0}^{\infty} E_i$

(D2) $Cn_R(\bigcup_{i=0}^{\infty} E_i) = \bigcup_{i=0}^{\infty} E_i$

(D3) $Cn_{D(E)}(\bigcup_{i=0}^{\infty} E_i) = \bigcup_{i=0}^{\infty} E_i$

By the minimality of $\Gamma_\Delta(E)$, we have $\Gamma_\Delta(E) \subseteq \bigcup_{i=0}^{\infty} E_i$.

Iterative Characterization of Extensions

Proof (Continued).

($\Rightarrow$) Let E be a default extension of Δ, i.e. $\Gamma_\Delta(E) = E$. We show that $E = \bigcup_{i=0}^{\infty} E_i$.

(a) $E = \Gamma_\Delta(E) \subseteq \bigcup_{i=0}^{\infty} E_i$

(b) We inductively show that $\bigcup_{i=0}^{\infty} E_i \subseteq \Gamma_\Delta(E) = E$.

  (1) $E_0 = W \subseteq E$

  (2) Let $E_i \subseteq E$. We show that $E_{i+1} \subseteq E$.
  Consider $\phi \in E_{i+1}$.

   (i) If $\phi \in Cn_R(E_i)$ then $\phi \in E$ since $Cn_R(E) = E$.

   (ii) Let $d \in D(E)$ be a rule with $cons(d) = \phi$ and $pre(d) \subseteq E_i$.
   Since $E_i \subseteq E$, $pre(d) \subseteq E$. Then $\phi \in E$ since $Cn_{D(E)}(E) = E$.

Proof (Continued).

($\Leftarrow$) Let $E = \bigcup_{i=0}^{\infty} E_i$. We show that E is a default extension of Δ, i.e. $\Gamma_\Delta(E) = E$.

(a) $\Gamma_\Delta(E) \subseteq \bigcup_{i=0}^{\infty} E_i = E$

(b) We inductively show that $E = \bigcup_{i=0}^{\infty} E_i \subseteq \Gamma_\Delta(E)$.

(1) $E_0 = W \subseteq \Gamma_\Delta(E)$

(2) Let $E_i \subseteq \Gamma_\Delta(E)$. We show that $E_{i+1} \subseteq \Gamma_\Delta(E)$. Consider $\phi \in E_{i+1}$.

(i) If $\phi \in Cn_R(E_i)$ then $\phi \in Cn_R(\Gamma_\Delta(E)) = \Gamma_\Delta(E)$ since $E_i \subseteq \Gamma_\Delta(E)$.

(ii) Let $d \in D(E)$ be a rule with $cons(d) = \phi$ and $pre(d) \subseteq E_i$. Since $E_i \subseteq \Gamma_\Delta(E)$, $pre(d) \subseteq \Gamma_\Delta(E)$. Then $\phi \in \Gamma_\Delta(E)$ since $Cn_{D(E)}(\Gamma_\Delta(E)) = \Gamma_\Delta(E)$. □

Theorem

A default theory $\Delta = (D, W)$ *has an inconsistent extension iff* W *is inconsistent.*

Proof.

($\Rightarrow$) If E is an inconsistent extension then $E = \mathcal{L}$ since $Cn_R(E) = E$. Then $D(E) = \emptyset$ and $E_i = Cn_R(W)$ for each $0 < i$. We have $Cn_R(W) = \mathcal{L}$, i.e. W is inconsistent.

($\Leftarrow$) If W is inconsistent then $Cn_R(W) = \mathcal{L}$. Since $W \subseteq E$, $Cn_R(W) \subseteq Cn_R(E) \subseteq \mathcal{L}$, i.e. E is inconsistent. □

Corollary

If a default theory has an inconsistent extension then this is its only extension.

Minimality of Extensions

Theorem

Default extensions are minimal, i.e. if $E \subseteq F$ are default extensions then $E = F$.

Proof.

By the iterative characterization, $E = \bigcup_{i=0}^{\infty}$, $F = \bigcup_{i=0}^{\infty}$. We inductively prove that $F_i \subseteq E_i$ for all $0 \leq i$, i.e. $F \subseteq E$.

(1) Trivially $F_0 \subseteq E_0$.

(2) Let $F_i \subseteq E_i$. We show that $F_{i+1} \subseteq E_{i+1}$. Consider $\phi \in F_{i+1}$.

- (i) If $\phi \in Cn_R(F_i)$ then $\phi \in Cn_R(E_i) \subseteq E_{i+1}$ since $F_i \subseteq E_i$.
- (ii) Let $d \in D(F)$ be a rule with $cons(d) = \phi$ and $pre(d) \subseteq F_i$. We have $pre(d) \subseteq E_i$ since $F_i \subseteq E_i$ and $d \in D(E)$ since $E \subseteq F$. Hence $\phi \in E_{i+1}$.

□

Normal Default Theory

Exceptions in Default Logic: $$\frac{bird(X) : fly(X)}{fly(X)}$$

Closed World Assumption: $$\frac{: \neg employee(X)}{\neg employee(X)}$$

Frame Problem: $$\frac{holds(F, S), action(A) : holds(F, do(A, S))}{holds(F, do(A, S))}$$

Definition (Normal Default Theory)

A default d is *normal* iff $just(d) = \{cons(d)\}$. A default theory $\Delta = (D, W)$ is *normal* iff every default in D is normal.

Existence of Extensions

Theorem

Every normal default theory has an extension.

Proof.

Let $\Delta = (D, W)$ be a normal default theory. If W is inconsistent, then $\mathcal{L}$ is an extension of Δ.
Let W be consistent and $E = \bigcup_{i=0}^{\infty} E_i$ where

$$E_0 = W$$
$$E_{i+1} = Cn_R(E_i) \cup T_i$$

We choose maximal T_i such that

- $E_i \cup T_i$ is consistent, and
- $T_i \subseteq \{cons(d) \mid d \in D, pre(d) \subseteq E_i\}$.

Proof (Continued).

Let $T_i' = \{cons(d) \mid d \in D(E), pre(d) \subseteq E_i\}$. We show that $T_i = T_i'$, i.e. $E = \bigcup_{i=0}^{\infty} E_i$ is a default extension of Δ.

(a) We prove that $T_i \subseteq T_i'$.
Let $\phi \in T_i$ and $d \in D$ be a default with $cons(d) = \phi$ and $pre(d) \subseteq E_i$. Since $E_i \cup T_i$ is consistent, $\overline{\phi} \notin E$ and $\phi \in T_i'$.

(b) We prove by contradiction that $T_i' \subseteq T_i$.
Let $\phi \in T'_i \setminus T_i$ and $d \in D(E)$ be a rule with $cons(d) = \phi$ and $pre(d) \subseteq E_i$. Then $\overline{\phi} \notin E$. By the maximality of T_i, we have that $E_i \cup T_i \cup \{\phi\}$ is inconsistent, i.e. $Cn_R(E_i) \cup T_i \cup \{\phi\}$ is inconsistent, i.e. $E_{i+1} \cup \{\phi\}$ is inconsistent, i.e. $E \cup \{\phi\}$ is inconsistent since $E_{i+1} \subseteq E$. Then $\overline{\phi} \in Cn_R(E) = E$ and we have a contradiction. □