Lecture 7: Answer Set Programming

2-AIN-108 Computational Logic

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Example

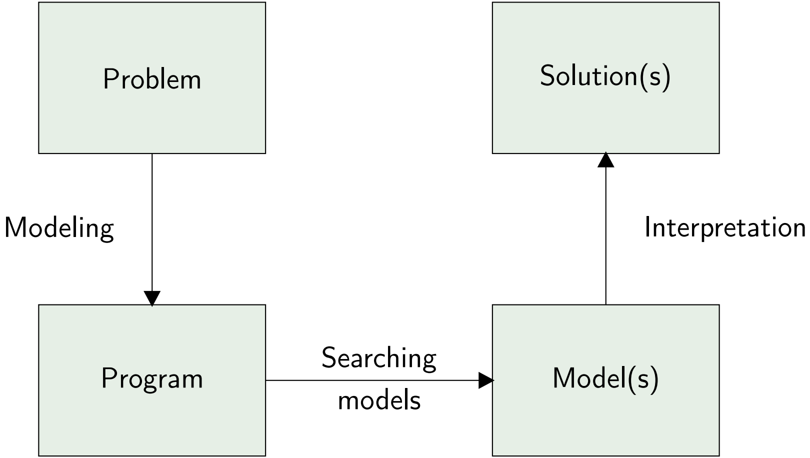
Logic Program:

$$
\begin{aligned}
father(abraham, isaac) &\leftarrow \\
mother(sarah, isaac) &\leftarrow \\
father(isaac, jacob) &\leftarrow \\
\\
parent(X, Y) &\leftarrow father(X, Y) \\
parent(X, Y) &\leftarrow mother(X, Y) \\
grandparent(X, Z) &\leftarrow parent(X, Y), parent(Y, Z) \\
ancestor(X, Y) &\leftarrow parent(X, Y) \\
ancestor(X, Z) &\leftarrow parent(X, Y), ancestor(Y, Z)
\end{aligned}
$$

Models:

$$M = \{parent(abraham, isaac), parent(sarah, isaac), \dots\}$$

Answer Set Programming



Definite Logic Programs

Definition (Stable Model)

An interpretation I is a stable model of a definite logic program P iff I is the least model of P.

Fact (Existence of Stable Model)

Each definite logic program has exactly one stable model.

$\sim p$ is true (p is false) by default unless we prove p.

Example (One Stable Model)

$$\begin{aligned} p &\leftarrow \\ r &\leftarrow p, \sim q \end{aligned}$$

Example (Two Stable Models)

$$\begin{aligned} p &\leftarrow \sim q \\ q &\leftarrow \sim p \end{aligned}$$

Example (No Stable Model)

$$p \leftarrow \sim p$$

Definition (Program Reduct)

Let I be an interpretation. A program reduct of a normal logic program P is a definite logic program P^I obtained from P by

- deleting all rules with a default literal L in the body not satisfied by I
- deleting all default literals from remaining rules.

Definition (Stable Model)

An interpretation I is a stable model of a normal logic program P iff I is the least model of the program reduct P^I.

Fact (Existence of Stable Model)

A normal logic program may have zero, one, or multiple stable models.

Properties

Theorem

Stable model of a normal logic program P is a model of P.

Theorem

Stable model of a normal logic program P is a minimal model of P.

Definition (Support)

A normal rule $A \leftarrow A_1, \ldots, A_m, \sim A_{m+1}, \ldots, \sim A_n$ supports an atom A (w.r.t. an interpretation I) iff $\{A_1, \ldots, A_m\} \subseteq I$ and $\{A_{m+1}, \ldots, A_n\} \cap I = \emptyset$.
An interpretation I is supported by a normal logic program P iff for each atom A in I there exists a rule r in P supporting A.

Theorem

Stable model of a normal logic program P is supported by P.

Answer Set Programming and Completion

Theorem

Each stable model of a normal logic program P is a model of $Comp(P)$.

Example

Logic Program P:

$$\begin{array}{ccc} p & \leftarrow & q \\ q & \leftarrow & p \end{array}$$

Completion $Comp(P)$:

$$p \quad \leftrightarrow \quad q$$

$\{p, q\}$ is a model of $Comp(P)$ but it is not a stable model of P.

Answer Set Programming and Completion

Definition (Tight Logic Program)

A normal logic program P is tight if there exists a mapping ℓ from the Herbrand base $\mathcal{B}$ to the set of natural numbers $\mathbb{N}$ such that for each rule $A \leftarrow A_1, \dots, A_m, \sim A_{m+1}, \dots, \sim A_n$ in P and each $1 \leq i \leq m$ holds $\ell(A) > \ell(A_i)$.

Theorem

Let P be a tight normal logic program. A model of $Comp(P)$ is a stable model of P.

How we can compute stable models?

Definition (Four-Valued Interpretation)

A (four-valued) interpretation is a pair (T, F). An interpretation is consistent if $T \cap F = \emptyset$. A consistent interpretation is total if $T \cup F = \mathcal{B}$ (Herbrand base), otherwise it is partial.

Definition (Applicable Rule)

A normal rule $A \leftarrow A_1, \ldots, A_m, \sim A_{m+1}, \ldots, \sim A_n$ is applicable w.r.t. an interpretation (T, F) iff

- $\{A_1, \ldots, A_m\} \subseteq T$
- $\{A_{m+1}, \ldots, A_n\} \subseteq F$
- $A \notin T$

How we can compute stable models?

Input: Grounded normal logic program P.
Output: Stable model of P.

1. Start with the empty interpretation $(\emptyset, \emptyset)$.
2. Apply all applicable rules. If an inconsistency is derived, backtrack.
3. If there exists a rule containing some default assumptions with unknown value, but all other assumptions are true, assume they are false and go to 2. Otherwise go to 4.
4. Assume all atoms with unknown value are false. The resulting interpretation is a stable model of P.

Example

Example 1:

$$\begin{aligned} p &\leftarrow \\ r &\leftarrow p, q \\ s &\leftarrow p, \sim q \end{aligned}$$

Example 2:

$$\begin{aligned} p &\leftarrow \sim q \\ q &\leftarrow \sim p \\ q &\leftarrow p \end{aligned}$$

Positive Logic Programs

Definition (Positive Rule)

A positive rule is a rule of the form

$$A_0 \vee \cdots \vee A_m \leftarrow A_{m+1} \wedge \cdots \wedge A_n$$

where $0 \leq m \leq n$ and each A_i, $0 \leq i \leq n$, is an atom.

Definition (Positive Logic Program)

A logic progam is positive if it contains only positive rules.

Example (Positive Logic Program)

$$p \vee q \quad \leftarrow$$

Definition (Stable Model)

An interpretation I is a stable model of a positive logic program P iff I is a minimal model of P.

Fact (Existence of Stable Model)

Each positive logic program has at least one stable model.

Example (Many Stable Models)

The positive logic progam $P = \{p \vee q \leftarrow\}$ has two stable models $M_1 = \{p\}$ and $M_2 = \{q\}$.

Disjunctive Logic Programs

Definition (Disjunctive Rule)

A disjunctive rule is a rule of the form

$$A_0 \vee \cdots \vee A_m \leftarrow L_{m+1} \wedge \cdots \wedge L_n$$

where $0 \leq m \leq n$, each A_i, $0 \leq i \leq n$, is an atom, and each L_i, $m < i \leq n$, is a literal.

Definition (Disjunctive Logic Program)

A logic program is disjunctive if it contains only disjunctive rules.

Definition (Program Reduct)

Let I be an interpretation. A program reduct of a disjunctive logic program P is a positive logic program P^I obtained from P by

- deleting all rules with default literal L in the body not satisfied by I
- deleting all default literals from remaining rules.

Definition (Stable Model)

An interpretation I is a stable model of a disjunctive logic program P iff I is a minimal model of the program reduct P^I.

Fact (Existence of Stable Model)

A normal logic program may have zero, one, or multiple stable models.

Definition (Constraint)

A constraint is a rule of the form

$$\leftarrow L_1 \wedge \cdots \wedge L_n$$

where $0 \leq n$ and each L_i, $1 \leq i \leq n$, is a literal.

Definition (Stable Model)

An interpretation I is a stable model of a disjunctive logic program P with a set of constraints C iff I is a stable model of P and satisfies C.

Sudoku

- each cell contains at least one number

$$s(A, B, X, Y, 1) \vee \cdots \vee s(A, B, X, Y, 9) \leftarrow \\ d(A) \wedge d(B) \wedge d(X) \wedge d(Y)$$

- each number appears at most once in each column

$$\leftarrow s(A_1, B, X_1, Y, N) \wedge s(A_2, B, N_2, Y, N) \wedge (A_1, X_1) < (A_2, X_2)$$

- each number appears at most once in each row

$$\leftarrow s(A, B_1, X, Y_1, N) \wedge s(A, B_2, N, Y_2, N) \wedge (B_1, Y_1) < (B_2, Y_2)$$

- each number appears at most once in each box

$$\leftarrow s(A, B, X_1, Y_1, N) \wedge s(A, B, X_2, Y_2, N) \wedge (X_1, Y_1) < (X_2, Y_2)$$

Explicit Negation

$$\underbrace{\mathit{train} \qquad \rlap{$\overbrace{\emptyset \qquad \neg\, \mathit{train}}^{\sim \mathit{train}}$}}_{\sim \neg\, \mathit{train}}\phantom{\qquad \neg\, \mathit{train}}$$

$$\mathit{cross} \leftarrow \sim \mathit{train}$$

versus

$$\mathit{cross} \leftarrow \neg\, \mathit{train}$$

Extended Logic Programs

Definition (Literal)

A classical literal is an atom or an atom preceded by classical negation. A default literal is a classical literal preceded by default negation. A literal is either classical or default literal.

Definition (Extended Logic Program)

An extended logic program is a finite set of rules

$$L_0 \vee \cdots \vee L_m \leftarrow L_{m+1} \wedge \cdots \wedge L_n$$

where $0 \leq m \leq n$, each L_i, $0 \leq i \leq m$, is a classical literal, and each L_i, $m < i \leq n$, is a literal.

Definition (Herbrand Interpretation)

An extended Herbrand base is a set of ground classical literals. A set of classical literals is coherent if it does not contain an atom A and its classical negation $\neg A$. A Herbrand interpretation is a coherent subset of the extended Herbrand base.

Definition (Stable Model)

An interpretation I is a stable model of an extended logic program P iff I is a coherent stable model of the same logic program P with classical literals interpreted as new atoms.

Flying Penguins

$$
\begin{aligned}
fly(X) &\leftarrow bird(X) \wedge \sim\neg fly(X)\\
\neg fly(X) &\leftarrow penguin(X)\\
bird(X) &\leftarrow penguin(X)\\
bird(skippy) &\leftarrow\\
penguin(tweety) &\leftarrow
\end{aligned}
$$