Lecture 1: First-Order Logic

2-AIN-108 Computational Logic

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FOL: Syntax

Definition (Alphabet)

An alphabet contains

- Set of variables
  $V = \{x, y, z, \dots\}$
- Set of function symbols
  $F = \{f, g, h, \dots\}$
- Set of predicate symbols
  $P = \{p, q, r, \dots\}$
- Logical connectives
  $\neg, \lor, \land, \rightarrow, \leftrightarrow$
- Quantifiers
  $\forall \quad \exists$
- Auxiliary symbols
  $(\quad) \quad ,$

FOL: Syntax (cont.)

Definition (Arity)

Given an alphabet with function symbols F and predicate symbols P, arity is any function $arity\colon F \cup P \mapsto \mathbb{N}_0$.

Note:

- Arity specifies how many "arguments" each function and predicate required.
- Functions (predicates) of arity 0, 1, 2, 3, and so on are called: nullary, unary, binary, ternary, etc.
- Nullary predicates are also called ~~logical constants~~ propositional variables.
- Nullary functions are also called ~~constant terms~~ constants.

Definition (Term)

Given an alphabet and an arity function, a term is any of the following:

- a variable;
- a constant;
- an expression $f(t_1, \dots, t_n)$ if f is a function symbol with arity n and $t_1, \dots, t_n$ are terms.

Definition (Atom)

Given an alphabet and an arity function, an atom is an expression $p(t_1, \dots, t_n)$ where p is a predicate symbol with arity n and $t_1, \dots, t_n$ are terms.

Definition (Formulae)

Given an alphabet and an arity function, a formula is any expression of the following forms:

- an atom;
- $\neg\Phi$;
- $(\Phi \wedge \Psi)$;
- $(\Phi \vee \Psi)$;
- $(\Phi \to \Psi)$;
- $(\Phi \leftrightarrow \Psi)$;
- $(\forall x)\Phi$;
- $(\exists x)\Phi$;

where Φ, Ψ are formulae, and x is a variable.

Definition (Language of FOL)

The language of First Order Logic over some alphabet and the respective arity function is the set $\mathcal{L}$ of all formulae.

FOL: Syntax (cont.)

Note: from now on we will always assume some fixed FOL language $\mathcal{L}$ over some alphabet with the respective arity function.

FOL: Syntax (cont.)

Definition (Ground expressions)

A term, atom, or a formula is ground if it does not contain any variables.

Definition (Free vs. bounded variable occurrence)

An occurrence of some variable x in a formula Φ is free if it is not preceded by $(\exists x)$ nor by $(\forall x)$. The occurence is bounded otherwise.

Definition (Closed formulae)

A formula Φ is closed if it does not contain any free occurrence of any variable.

Note: from now on we will assume that all formulae are closed.

FOL: Syntax (cont.)

Definition (Theory)

A first order theory (or just theory) T is a finite set of (closed) formulae.

Note: we will look at theories as knowledge bases: a theory T is a set of formulae that describes some situation or some problem.

Example

Let us assume the following situation: *Jack killed John. If someone killed somebody else, he is a murderer. Murderers go to jail.* We may encode this in FOL theory T:

$$\mathrm{Killed}(\mathrm{Jack}, \mathrm{John})$$
$$(\forall x)((\exists y)\mathrm{Killed}(x, y) \rightarrow \mathrm{Murderer}(x))$$
$$(\forall x)(\mathrm{Murderer}(x) \rightarrow \mathrm{Jail}(x))$$

Definition (First order interpretations)

An ~~structure~~ interpretation is a pair ~~$\mathcal{D}$~~ $\mathcal{I} = (D, I)$ where

- D, called domain, is a nonempty set;
- I is an ~~interpretation~~ interpretation function s.t.:
  - $I(f)$ is a function $f^I : D^{arity(f)} \to D$;
  - $I(t)$ is $t^I = f^I(t_1^I, \dots, t_n^I)$ for any ground term of the form $t = f(t_1, \dots, t_n)$;
  - $I(p)$ is a relation $p^I \subseteq D^{arity(p)}$.

Note: $D^0 = \{\emptyset\}$, hence there are two possible interpretations of each propositional variable p: either $p^I = \{\emptyset\}$ (i.e., p is *true*) or $p^I = \emptyset$ (i.e., p is false).

Note: similarly for a constant c: $c^I : D^0 \to D$, i.e., each constant term is interpreted by a constant function which returns one of the elements of D.

FOL: Semantics (cont.)

Definition (Interpretation extension)

An extension of an interpretation $\mathcal{I} = (D, I)$ w.r.t. a variable x is an interpretation $\mathcal{I}' = (D, I')$ where I' is identical to I except for in addition $I'(x) = d$ for some element $d \in D$.

Definition (Satisfaction $\models$)

A formula Π is satisfied w.r.t. an interpretation $\mathcal{I}$ (denoted by $\mathcal{I} \models \Phi$) based type of Π:

$p(t_1, \ldots, t_n)$: $\mathcal{I} \models p(t_1, \ldots, t_n)$ iff $(t_1^I, \ldots, t_n^I) \in p^I$;

$\neg\Phi$: $\mathcal{I} \models \neg\Phi$ iff $\mathcal{I} \not\models \Phi$;

$\Phi \wedge \Psi$: $\mathcal{I} \models (\Phi \wedge \Psi)$ iff $\mathcal{I} \models \Phi$ and $\mathcal{I} \models \Psi$;

if $\Phi \vee \Psi$: $\mathcal{I} \models (\Phi \vee \Psi)$ iff $\mathcal{I} \models \Phi$ or $\mathcal{I} \models \Psi$;

$\Phi \rightarrow \Psi$: $\mathcal{I} \models (\Phi \rightarrow \Psi)$ iff $\mathcal{I} \not\models \Phi$ or $\mathcal{I} \models \Psi$;

$\Phi \leftrightarrow \Psi$: $\mathcal{I} \models (\Phi \leftrightarrow \Psi)$ iff $(\mathcal{I} \models \Phi$ iff $\mathcal{I} \models \Psi)$;

$(\exists x)\Phi$: $\mathcal{I} \models (\exists x)\Phi$ iff $\mathcal{D}' \models \Phi$ for some ext. $\mathcal{I}'$ of $\mathcal{I}$ w.r.t. x;

$(\forall x)\Phi$: $\mathcal{I} \models (\forall x)\Phi$ iff $\mathcal{D}' \models \Phi$ for all ext. $\mathcal{I}'$ of $\mathcal{I}$ w.r.t. x;

where Φ, Ψ are any formulae and $p(t_1, \ldots, t_n)$ is any ground atom.

Semantics (cont.)

Definition (Model)

An interpretation $\mathcal{I}$ is a model of Φ if $\mathcal{I} \models \Phi$; $\mathcal{I}$ is a model of a theory T (denoted $\mathcal{I} \models T$) if $\mathcal{I} \models \Phi$ for all $\Phi \in T$.

Definition (Satisfiability)

A formula (or theory) is satisfiable, if it has a model.

Definition (Entailment)

A theory T entails a formula Φ (denoted $T \models \Phi$) if for each model $\mathcal{I}$ of T we have $\mathcal{I} \models \Phi$.

Example (cont.)

Is there a model of our theory T? T was:

$$\text{Killed}(\text{Jack}, \text{John})$$
$$(\forall x)((\exists y)\text{Killed}(x, y) \rightarrow \text{Murderer}(x))$$
$$(\forall x)(\text{Murderer}(x) \rightarrow \text{Jail}(x))$$

Example (cont.)

Is there a model of our theory T? T was:

$$\mathrm{Killed}(\mathrm{Jack}, \mathrm{John})$$
$$(\forall x)((\exists y)\mathrm{Killed}(x, y) \to \mathrm{Murderer}(x))$$
$$(\forall x)(\mathrm{Murderer}(x) \to \mathrm{Jail}(x))$$

Let us construct $\mathcal{I} = (\{s\}, I)$ with:

$$\begin{aligned} \mathrm{Jack}^I &= s \\ \mathrm{John}^I &= s \\ \mathrm{Killed}^I &= \{\langle s, s\rangle\} \\ \mathrm{Murderer}^I &= \{\langle s\rangle\} \\ \mathrm{Jail}^I &= \{\langle s\rangle\} \end{aligned}$$

Example (cont.)

Is there a model of our theory T? T was:

$$\text{Killed}(\text{Jack}, \text{John})$$
$$(\forall \text{x})((\exists \text{y})\text{Killed}(\text{x}, \text{y}) \rightarrow \text{Murderer}(\text{x}))$$
$$(\forall \text{x})(\text{Murderer}(\text{x}) \rightarrow \text{Jail}(\text{x}))$$

Let us construct $\mathcal{I} = (\{s\}, I)$ with:

$$\begin{aligned}
\text{Jack}^I &= s \\
\text{John}^I &= s \\
\text{Killed}^I &= \{\langle s, s\rangle\} \\
\text{Murderer}^I &= \{\langle s\rangle\} \\
\text{Jail}^I &= \{\langle s\rangle\}
\end{aligned}$$

Is $\mathcal{I}$ a model of T?

Example (cont.)

Is there a model of our theory T? T was:

$$\text{Killed}(\text{Jack}, \text{John})$$
$$(\forall x)((\exists y)\text{Killed}(x, y) \rightarrow \text{Murderer}(x))$$
$$(\forall x)(\text{Murderer}(x) \rightarrow \text{Jail}(x))$$

Let us construct $\mathcal{I} = (\{s\}, I)$ with:

$$\text{Jack}^I = s$$
$$\text{John}^I = s$$
$$\text{Killed}^I = \{\langle s, s\rangle\}$$
$$\text{Murderer}^I = \{\langle s\rangle\}$$
$$\text{Jail}^I = \{\langle s\rangle\}$$

Is is our indented model of T?

Example (cont.)

Is there a model of our theory T? T was:

$$\begin{gathered}\text{Killed}(\text{Jack}, \text{John}) \\ (\forall x)((\exists y)\text{Killed}(x, y) \rightarrow \text{Murderer}(x)) \\ (\forall x)(\text{Murderer}(x) \rightarrow \text{Jail}(x))\end{gathered}$$

Let us construct $\mathcal{I} = (\{s\}, I)$ with:

$$\begin{aligned}\text{Jack}^I &= s \\ \text{John}^I &= s \\ \text{Killed}^I &= \{\langle s, s\rangle\} \\ \text{Murderer}^I &= \{\langle s\rangle\} \\ \text{Jail}^I &= \{\langle s\rangle\}\end{aligned}$$

Does if holds $T \models \text{Murderer}(\text{Jack})$?